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Knot Invariants Associated with a Particular $N \to \infty$ Continuous Limit of the Baxter–Bazhanov Model

B. L. Cerchiai,[1] P. Cotta-Ramusino,[2] and M. Martellini[3]

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First we briefly recall the definition of the three-dimensional Baxter–Bazhanov lattice model. The spins of this model are elements of $\mathbf{Z}_N$ and the R-matrix is associated to the algebra $U_q sl(n)$ if q is a primitive Nth root of unity. Then we construct a particular $N \to \infty$ limit of the model, in which it is meaningful to interpret the spins as elements of $\mathbf{R}$ and which gives the free Gaussian boson model. Finally, we study special limits of the rapidity variables in which we obtain braid group representations and we show that for n odd the associated knot invariants are given by the inverse of products of Alexander polynomials, evaluated at certain roots of unity.

KEY WORDS: Three-dimensional solvable models; tetrahedron equation; Baxter–Bazhanov models; generalized chiral Potts models; Yang–Baxter equation; braid group representations; Alexander knot invariants.

1. INTRODUCTION

Baxter and Bazhanov[1] introduced an integrable model on a cubic lattice which is particularly interesting because it is one of the few solvable models in three dimensions. It is an IRF model and the spins are elements of $\mathbf{Z}_N$.

The Boltzmann weights of the Baxter–Bazhanov model give a solution of the tetrahedron equations which is a generalization of Zamolodchikov's.[2,3] A very important feature of the Baxter–Bazhanov model is that, apart from a slight modification of the boundary conditions, its two-dimensional

[1] DAAD-Fellow, Sektion Theoretische Physik, Lehrstuhl Wess, Ludwig-Maximilians-Universität, 80333 Munich, Germany. E-mail: bianca@lswes8.ls-wess.physik.uni-muenchen.de.

[2] Dipartimento di Matematica, Università di Milano, Milan, Italy, and INFN, Sezione di Milano, Italy. E-mail: cotta@vaxmi.mi.infn.it.

[3] Dipartimento di Fisica, Università di Milano, Milan, Italy, and INFN, Sezione di Pavia, 27100 Pavia, Italy. E-mail: martellini@vaxmi.mi.infn.it.

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reduction gives the chiral Potts model.[4,5] It is one of the most interesting integrable models and its Boltzmann weights satisfy the Yang–Baxter equation[5,6] and are related to a cyclic representation of the algebra $U_q(sl(n))$ for $q = e^{2\pi i/N}$.

In ref. 7 we proved that it is possible to find limits of the spectral parameters in which the Boltzmann weights of the Baxter–Bazhanov model give braid group representations and we calculated the associated (cyclotomic) knot invariants.[8–11]

In this paper we continue the study begun in ref. 7. In particular, we explore a large-N limit of the model. We show that making a rescaling of the spins $l/\sqrt{N} \to l$ in the limit $N \to \infty$, it is meaningful to interpret the state variables of the model as elements of $\mathbf{R}$. The model which is obtained with this continuation procedure is equivalent to the free Gaussian boson model introduced in ref. 12.

Further, we see that for special limits of the rapidity variables it is possible to obtain (infinite-dimensional) braid group representations from the Boltzmann weights of the free boson model. In these limits we perform directly the Gaussian integration which gives the associated knot invariants and we show that for n odd they can be expressed as the inverse of a product of Alexander polynomials. As a consequence, by Milnor's theorem,[13] the partition function $\mathscr{Z}_\infty$ of the model is proportional to the Reidemeister–Ray–Singer torsion for a specific manifold.

In addition, the Alexander polynomials obtained in this paper are evaluated at the nth roots of unity, and these values are particularly interesting. From the point of view of the homology theory, as in the N-finite case, the invariants are connected with a presentation matrix of the Abelian group $H_1(\Sigma_n, \mathbf{Z})$ if Σ_n is the nth cyclic covering space of S^3 branched along the link.

2. THE BAXTER–BAZHANOV MODEL AND ITS TWO-DIMENSIONAL REDUCTION

The Baxter–Bazhanov model[1] is a three-dimensional integrable IRF model in which a state variable is associated with each site of a three-dimensional cubic lattice. The model depends on an integer N which fixes the number of values that a single spin can take. In fact, the state variables are elements of $\mathbf{Z}_N$. This is one of the most important features of the model. The Baxter–Bazhanov model is a generalization of the Zamolodchikov model,[2,3] the first three-dimensional integrable model, which was introduced at the beginning of the 1980s and which can be recovered when $N = 2$.

In order to define the Boltzmann weight of an elementary cube of the Baxter–Bazhanov model it is necessary to fix some notation first. Let

$$\omega = e^{2\pi i/N} \tag{2.1}$$

be a primitive Nth root of the unity, and

$$\omega^{1/2} = e^{\pi i/N} \tag{2.2}$$

be its square root. Let $\Phi(l)$ and $s(k, l)$ be functions such that

$$\Phi(l) = (\omega^{1/2})^{l(l+N)} \tag{2.3}$$

$$s(k, l) = \omega^{kl} \tag{2.4}$$

and $w(x, l)$ be the function such that

$$w(x, l) = \Delta(x)^l \prod_{k=1}^{l} (1 - \omega^k x)^{-1} \tag{2.5}$$

where x is a complex number, k, l are integers, and $\Delta(x)$ is given by

$$\Delta(x)^N = (1 - x^N) \tag{2.6}$$

Let us fix the four rapidity variables $p, p', q, q' \in \mathbf{C}$. These variables are temperature-like variables. The Boltzmann weight of the elementary cube of the Baxter–Bazhanov model is defined by

$$W(a\,|\,e, f, g\,|\,b, c, d\,|\,h) = \sum_{\sigma=0}^{N-1} v_\sigma(a\,|\,e, f, g\,|\,b, c, d\,|\,h) \tag{2.7}$$

where

$$\begin{aligned} &v_\sigma(a\,|\,e, f, g\,|\,b, c, d\,|\,h) \\ &\quad = \frac{w(p'/p, e-c-d+h)}{w(p'/p, a-g-f+b)}\, s(c-h, d-h)\, s(g, a-g-f+b) \\ &\quad\quad \times \left\{ \frac{w(p/q, d-h-\sigma)\, w(q'/p, \sigma-f+b)\, w(p'/q', a-g-\sigma)}{w(p'/q, e-c-\sigma)(\Phi(a-g-\sigma))^{-1}} \right. \\ &\quad\quad \left. \times\, s(\sigma, a-c-f+h) \right\} \end{aligned} \tag{2.8}$$

The spins a,..., h are placed at the eight vertices of the elementary cell as is shown in Fig. 1.

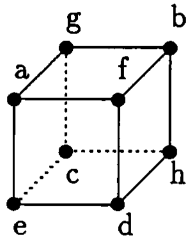
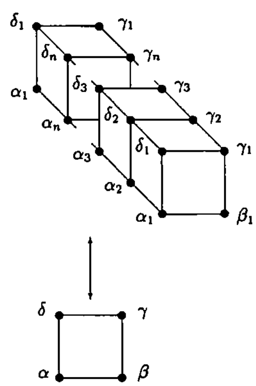


Fig. 1. Elementary cube of the Baxter–Bazhanov model.

It is possible to connect the Baxter–Bazhanov model with a two-dimensional model through the following reduction procedure (Fig. 2). We choose one of the three dimensions in space with periodic boundary conditions. Let there be n elementary cubes in that direction. It is possible to map all the n spins located along a line in that direction in a single spin with n components. This new state variable is placed at a vertex of a new lattice in such a way that the whole line of the three-dimensional lattice becomes one point of the two-dimensional lattice. With this technique it is possible to reduce the three-dimensional Baxter–Bazhanov model to the two-dimensional chiral Potts model, apart from a slight change of the boundary conditions. In this context it is meaningful to consider a whole parallelepiped $\mathscr{P}$ formed by a row of n cubes and to study its Boltzmann weight. The spins are written in the form $\alpha = (\alpha_1, ..., \alpha_n)$, where $\alpha_i \in \mathbf{Z}_N\ \forall i$. It



$\alpha = (\alpha_1, \dots, \alpha_n), \dots, \delta = (\delta_1, \dots, \delta_n)$

Fig. 2. Reduction procedure: a parallepiped $\mathscr{P}$ of the three-dimensional Baxter–Bazhanov model becomes an elementary square of the chiral Potts model.

is important to notice that the Boltzmann weight is defined in such a way that the following equivalence relation holds:

$$\alpha \sim \beta \Leftrightarrow \alpha_i - \alpha_{i+1} = \beta_i - \beta_{i+1} \qquad \forall i = 1, \ldots, n \tag{2.9}$$

This means that the Boltzmann weight is not modified if a spin α is replaced by another spin β which is equivalent to α according to the equivalence relation (2.9). This is a technique to implement the $\mathbf{Z}_N^{n-1}$ symmetry of the model, which is another of its main features.

The Boltzmann weight of the whole row of n cubes of the Baxter–Bazhanov model is given by the following expression:

$$S(\alpha, \beta, \gamma, \delta) = \prod_{i \in \mathbf{Z}_n} W(\delta_i \mid \alpha_i, \gamma_i, \delta_{i+1} \mid \gamma_{i+1}, \alpha_{i+1}, \beta_i \mid \beta_{i+1}) \tag{2.10}$$

Making the change of the boundary conditions

$$\sigma_i = \mu_i - \mu_{i+1} \tag{2.11}$$

we obtain a modified Baxter–Bazhanov model, and $\mathscr{P}$ has a Boltzmann weight which is defined by

$$S_0(\alpha, \beta, \gamma, \delta) = \prod_{i \in \mathbf{Z}_n} \sum_{\mu_i} v_{\mu_i - \mu_{i+1}}(\delta_i \mid \alpha_i, \gamma_i, \delta_{i+1} \mid \gamma_{i+1}, \alpha_{i+1}, \beta_i \mid \beta_{i+1}) \tag{2.12}$$

The connection with the chiral Potts model[4,5] arises because the Boltzmann weight of an elementary cell of that model is given exactly by the expression (2.12), which in fact can be written in the form

$$\begin{aligned} S_0(p, p', q, q' \mid \alpha, \beta, \gamma, \delta) \\ = \sum_{\mu} \frac{W_{p'p}(\alpha, \beta)}{W_{p'p}(\delta, \gamma)} \frac{W_{pq}(\beta, \mu)\, W_{q'p}(\mu, \gamma)\, W_{p'q'}(\delta, \mu)}{W_{p'q}(\alpha, \mu)} \end{aligned} \tag{2.13}$$

where $W_{pq}(\alpha, \beta)$ is given by

$$W_{pq}(\alpha, \beta) = \prod_{i=1}^{n} \left[\omega^{(\beta_i - \beta_{i+1})(\alpha_{i+1} - \beta_{i+1})} w\left(\frac{p_i}{q_i}, \alpha_i - \alpha_{i+1} - \beta_i + \beta_{i+1}\right)\right] \tag{2.14}$$

The Baxter–Bazhanov model is integrable and the Boltzmann weight S introduced in (2.7) is a solution of the tetrahedron equations.[14–16] From this it follows that the Boltzmann weight S_0, (2.12), of the chiral Potts model is a solution of the Yang–Baxter equations[5,6]

$$\begin{aligned}\sum_{\sigma} S_0(p, p', q, q' \mid \alpha, \beta, \gamma, \sigma)\, S_0(p, p', r, r' \mid \sigma, \gamma, \delta, \varepsilon)&\\ \times S_0(q, q', r, r' \mid \alpha, \sigma, \varepsilon, \kappa)&\\ =\sum_{\sigma} S_0(q, q', r, r' \mid \beta, \gamma, \delta, \sigma)&\\ \times S_0(p, p', r, r' \mid \alpha, \beta, \sigma, \kappa)\, S_0(p, p', q, q' \mid \kappa, \sigma, \delta, \varepsilon)& \qquad (2.15)\end{aligned}$$

The same holds for S:

$$\begin{aligned}\sum_{\sigma} S(p, p', q, q' \mid \alpha, \beta, \gamma, \sigma)\, S(p, p', r, r' \mid \sigma, \gamma, \delta, \varepsilon)&\\ \times S(q, q', r, r' \mid \alpha, \sigma, \varepsilon, \kappa)&\\ =\sum_{\sigma} S(q, q', r, r' \mid \beta, \gamma, \delta, \sigma)&\\ \times S(p, p', r, r' \mid \alpha, \beta, \sigma, \kappa)\, S(p, p', q, q' \mid \kappa, \sigma, \delta, \varepsilon)& \qquad (2.16)\end{aligned}$$

3. THE LIMIT $N \to \infty$

In this section we study a particular continuous limit for $N \to \infty$ of the Boltzmann weights $S_0(\alpha, \beta, \gamma, \delta)$ of the modified Baxter–Bazhanov model. We restrict ourselves to the trigonometric case, in which

$$p_i = p_j, \qquad q_i = q_j, \qquad p'_i = p'_j, \qquad q'_i = q'_j \qquad \forall i, j \in Z_n \qquad (3.1)$$

On the one hand this means that the model is homogeneous, and that all the cubes have the same rapidity variables. On the other hand it means also that the model is critical and this is an element which is important for obtaining topological invariants.

To study the limit $N \to \infty$, the first thing is to calculate the expression of the function $w(x, l)$ defined in (2.5),

$$w(x, l) = \Delta(x)^l \prod_{k=1}^{l} (1 - \omega^k x)^{-1}$$

We obtain in the limit $N \to \infty$

$$
\begin{aligned}
w(x,l) &= \prod_{k=1}^{N}\left[(1-\omega^k x)^{l/N}\right]\prod_{k=1}^{l}\left[(1-\omega^k x)^{-1}\right] \\
&= \exp\left\{\frac{l}{N}\sum_{k=1}^{N}\left[\log(1-\omega^k x)\right]-\sum_{k=1}^{l}\left[\log(1-\omega^k x)\right]\right\} \\
&\to \exp\left(\frac{l}{N}\sum_{k=1}^{N}\left\{\log\left[1-\left(1+2\pi i\frac{k}{N}\right)x\right]\right\}\right. \\
&\qquad \left.-\sum_{k=1}^{l}\left\{\log\left[1-\left(1+2\pi i\frac{k}{N}\right)x\right]\right\}\right) \\
&= \exp\left\{\frac{l}{N}\sum_{k=1}^{N}\left[\log(1-x)+\log\left(1-2\pi i\frac{k}{N}\frac{x}{1-x}\right)\right]\right\} \\
&\quad \times \exp\left\{-\sum_{k=1}^{l}\left[\log(1-x)+\log\left(1-2\pi i\frac{k}{N}\frac{1}{1-x}\right)\right]\right\} \\
&\to \exp\left[\sum_{k=1}^{N}\left(-\frac{l}{N^2}2\pi ik\frac{x}{1-x}\right)+\sum_{k=1}^{l}\left(2\pi i\frac{k}{N}\frac{x}{1-x}\right)\right] \\
&= \exp\left[-\frac{l}{N^2}\pi iN(N+1)\frac{x}{1-x}+\pi i\frac{l(l+1)}{N}\frac{x}{1-x}\right] \\
&= \exp\left[\pi i\frac{l(N-l)}{N}\frac{x}{x-1}\right]
\end{aligned} \tag{3.2}
$$

As a consequence of the definition (2.12) of the model the terms in the exponent which are linear in l cancel out. This is a consequence of the fact that these terms give expressions of the type $\sum_{i=1}^{n}\alpha_i-\alpha_{i+1}=\alpha_1-\alpha_{n+1}=0$ in the case of periodic boundary conditions. There remains

$$
w(x,l)\to\exp\left(-\pi i\frac{l^2}{N}\frac{x}{x-1}\right)\qquad\text{for}\quad N\to\infty \tag{3.3}
$$

It is meaningful to consider l as an element of **R**. This means that we calculate the limit of the Boltzmann weight S_0 in the case $N\to\infty$ and then we interpret $l=: l/\sqrt{N}$ as a new continuous variable, taking values in **R**. Thus, we define

$$
w(x,l)\to\exp\left(\pi il^2\frac{x}{1-x}\right)\qquad\text{with}\quad l\in\mathbf{R} \tag{3.4}
$$

with $l\in\mathbf{R}$. Further, in an analogous way, we define

$$
\phi(l)=e^{\pi il^2} \tag{3.5}
$$

and

$$s(k, l) = e^{2\pi i k l} \tag{3.6}$$

where k, $l \in \mathbf{R}$. With these notations we can reconstruct S_0 as in Eqs. (2.12) and (2.8) by substituting the sums with integrals. Notice that in the continuous limit it is formally correct to put $N=1$ in the equations defining the Boltzmann weights and that

$$w(x, 0) = 1 \tag{3.7}$$

At this point some observations are in order.

First, it should be noticed that the model defined by Eqs. (3.4)–(3.6) is equivalent to the free boson model found by Baxter and Bazhanov in ref. 12. In fact it is possible to obtain the free boson model of ref. 12 by the transformation of the spectral parameters given by

$$\frac{v}{1-v} = -2\pi \frac{x}{1-x} \tag{3.8}$$

where x is the rapidity variable appearing in Eq. (3.4) and v the rapidity variable in Eq. (4.1) of ref. 12. Thus, in the thermodynamic limit the logarithm of its partition function κ_B is given by Eq. (4.8) of ref. 12,

$$\log \kappa_B = -2 \log \kappa_2 \tag{3.9}$$

Further, it is very important to notice that Eqs. (3.4)–(3.6) imply a particular continuation procedure of the state variables. In fact, we obtain the Gaussian free boson model of ref. 12 because we have made a rescaling of the spins and we choose $l/\sqrt{N}$ as the new continuous variable. Thus the model constructed in this way is not equivalent to a model obtained in the large-N limit of the Baxter–Bazhanov model with a different continuation procedure, in particular with a different or no rescaling of the state variables.

This can be verified immediately. Equation (3.9) does not coincide with the result which is obtained by simply taking the N-infinite limit of the logarithm of the partition function κ_N of the finite-N Baxter–Bazhanov model in the thermodynamic limit, which is given by Eq. (3.34) of ref. 12,

$$\lim_{N\to\infty} \log \kappa_N = \lim_{N\to\infty} 2\frac{N-1}{N} \log \kappa_2 = 2 \log \kappa_2 \tag{3.10}$$

It results that $\log \kappa_B$ and the N-infinite limit of $\log \kappa_N$ differ by a sign.

This is a consequence of the fact that (3.10) defines an implicit procedure to calculate the limit $N \to \infty$ which is different from the procedure used to obtain (3.4)–(3.6) because no rescaling and no continuation of the spins to the real numbers is done. In fact, in (3.10) the thermodynamic limit is taken before the limit $N \to \infty$ is calculated. This means that the thermodynamic limit and the limit $N \to \infty$ are taken in a different order in (3.9) and (3.10), and the limits may not commute for the above-mentioned reasons.

Let us provide another example of continuous N-infinite limit of the Boltzmann weights S_0 of the Baxter–Bazhanov model. Let us consider the case $n=2$. In the trigonometric limit the $sl(2)$-chiral Potts model reduces to the Fateev–Zamolodchikov model.[19] In the limit $N \to \infty$ Fateev and Zamolodchikov obtain a model with a $U(1)$ symmetry.[19,20] The connection between the model which is studied here and the model studied by Fateev and Zamolodchikov in refs. 19 and 20 is that before calculating the limit $N \to \infty$ Fateev and Zamolodchikov apply the self-duality property of their model to make a Kramers–Wannier transform, which is a kind of Fourier transform. In this way they preserve the property of the spins to be cyclic and hence obtain the $U(1)$ symmetry, but they lose the self-duality. On the other hand with the method outlined in this section we lose the cyclicity of the spins, but the self-duality is maintained, because the Fourier transform of a Gaussian function remains a Gaussian function. This difference can be related to a different rescaling of the spin variable which is transformed to a real variable, namely Fateev and Zamolodchikov interpret the spin variable l/N as the new continuous variable, because N gives the period of the spins and they want to keep the model cyclic. Apart from this they also use a different parametrization of the spectral parameter. Their rapidity variable α is connected to the rapidity variable in (2.5) by the rule

$$e^{i\alpha/N}\omega^{-1/2} = x$$

It should also be noticed that the limits of $w(x, l)$ when $x \to 0$, 1, ∞ are exactly the same as are obtained in the case that N is finite,

$$w(x, l) \to \begin{cases} 1 & \text{if} \quad x \to 0 \\ \delta_{(l)} & \text{if} \quad x \to 1 \\ (\phi(l))^{-1} & \text{if} \quad x \to \infty \end{cases} \tag{3.11}$$

It can be verified by analogy with the case N finite that the infinite-dimensional model satisfies the Yang–Baxter equation

$$\int_{\sigma\in\mathbf{R}^n} d\sigma\, S_0(p,p',q,q' \mid \alpha,\beta,\gamma,\sigma)\, S_0(p,p',r,r' \mid \sigma,\gamma,\delta,\varepsilon)$$
$$\times S_0(q,q',r,r' \mid \alpha,\sigma,\varepsilon,\kappa)$$
$$=\int_{\sigma\in\mathbf{R}^n} d\sigma\, S_0(q,q',r,r' \mid \beta,\gamma,\delta,\sigma)$$
$$\times S_0(p,p',r,r' \mid \alpha,\beta,\sigma,\kappa)\, S_0(p,p',q,q' \mid \kappa,\sigma,\delta,\varepsilon) \qquad (3.12)$$

4. ALEXANDER INVARIANTS IN THE CASE $N \to \infty$

By applying (3.12) and (3.11), it is possible to search for the limits of the spectral parameters which give (infinite-dimensional) representations of the braid group and to calculate the corresponding values of the Boltzmann weight S_0. First of all we introduce the Yang–Baxter operators with matrix elements given by

$$\langle \alpha(1)\cdots\alpha(M-1) \mid Y_k(p,p',q,q') \mid \alpha'(1)\cdots\alpha'(M-1)\rangle$$
$$=\left[\prod_{l\neq k}\delta(\alpha(l)-\alpha'(l))\right] S_0(\alpha(k-1),\alpha(k),\alpha(k+1),\alpha'(k)) \qquad (4.1)$$

where M is a fixed integer. By choosing particular limits of the spectral parameters, the operators Y_k give a representation of the braid group on M strings B_M, even if it is infinite-dimensional, because the spins take values in $\mathbf{R}^n$. In the finite-dimensional, because the spins take values in $\mathbf{R}^n$. In the finite-dimensional case, given a knot, it is possible to obtain an associated invariant by a standard method,[17,18] namely by calculating the trace of an element of the braid group corresponding to a diagram of the knot in the representation spanned by the Yang–Baxter operators Y_k defined by (4.1). It can be observed that also in the continuous large-N limit the Yang–Baxter operators belong to an algebra similar to the cyclotomic one which is obtain in the case N finite.[8–11] Therefore it is possible to repeat the same procedure used in the finite-dimensional case to see that the trace in the above-mentioned braid group representations gives topological invariants, and that they are unitary representations.

If we want to eliminate the equivalence relation (2.9) the following change of basis in the spin set L can be made:

$$\alpha_i \to \alpha_i - \alpha_{i+1} \qquad (4.2)$$

for $i=1,\ldots,n-1$. In fact the generator α_n is superflous, because the spin set has the structure of a vector space and its dimension is $n-1$ and not n

because of the equivalence relation (2.9). As in the finite-dimensional case we introduce the quadratic form on the spin set defined through the $n-1 \times n-1$ matrix B

$$B_{ij} = \begin{cases} -1 & \text{if } \; i \leqslant j \\ 0 & \text{otherwise} \end{cases} \tag{4.3}$$

In this context it should be noticed that the relation

$$\sum_{\alpha \in L} \omega^{B(\alpha, \beta)} = N^{n-1} \delta_{\beta, 0} \tag{4.4}$$

which holds in the case N finite, becomes

$$\int_{\alpha \in L} d\alpha \, e^{2\pi i B(\alpha, \beta)} = \delta(\beta) \tag{4.5}$$

in the case N infinite.

Moreover, it can be verified that if we take, e.g., the following limit of the rapidity variables

$$\text{(Ia)} \quad p \ll q \ll p' = q' \tag{4.6}$$

and if we multiply the Boltzmann weight S_0 by the factor

$$\exp\{\pi i[B(\delta, \delta) - B(\beta, \beta) + B(\beta - \delta, \gamma) + B(\alpha, \beta - \delta)]\} \tag{4.7}$$

which is only an equivalence relation that does not modify the partition function or the commutation relation of the transfer matrices of the model, we obtain the following expression for the Yang–Baxter operators:

$$\begin{aligned} &\langle \alpha(1) \cdots \alpha(M-1) \,|\, Y_k \,|\, \alpha'(1) \cdots \alpha'(M-1) \rangle \\ &\quad = \left[\prod_{l \neq k} \delta(\alpha(l) - \alpha'(l)) \right] \exp\{2\pi i[B(\alpha'(k), \alpha(k)) \\ &\qquad - \tfrac{1}{2} B(\alpha'(k), \alpha'(k)) - \tfrac{1}{2} B(\alpha(k), \alpha(k)) \\ &\qquad - \tfrac{1}{2} B(\alpha(k-1), \alpha(k) - \alpha'(k)) + B(\alpha(k) - \alpha'(k), \alpha(k+1))]\} \end{aligned} \tag{4.8}$$

By using this formula it can be shown that the trace of an element t of the braid group which can be closed to give a knot K in the representation spanned by the operators Y_k can be written in the form

$$\text{Tr}(t) = \int_{\alpha \in R^{u} \otimes R^{n-1}} d\alpha \, e^{\pi i Q(\alpha, \alpha)} \tag{4.9}$$

where

$$Q = S \otimes B + S^T \otimes B^T \tag{4.10}$$

and S is a $v \times v$ Seifert matrix for the knot and T denotes the transposition of a matrix. In this expression the trace is defined as

$$\mathrm{Tr}(t) = \int_{\alpha \in L^{M-1}} d\alpha \, \langle \alpha \, | t | \, \alpha \rangle \tag{4.11}$$

It should be noticed that the expression (4.9) gives the partition function $\mathscr{Z}_\infty$ of the continuous large-N limit of the Baxter–Bazhanov model corresponding to the transfer matrix t.

To see that (4.9) effectively holds, the following procedure can be used. The operators Y_k in (4.8) in the preceding limit of the spectral parameters can be written in the form

$$\begin{aligned} &\langle \alpha(1) \cdots \alpha(M-1) \, | Y_k | \, \alpha'(1) \cdots \alpha'(M-1) \rangle \\ &\quad = \int_{\beta \in L} d\alpha \, e^{-\pi i B(\beta, \beta)} \langle \alpha(1) \cdots \alpha(M-1) \, | x_k^\beta | \, \alpha'(1) \cdots \alpha'(M-1) \rangle \end{aligned} \tag{4.12}$$

where x_k^β is the operator given by

$$\begin{aligned} &\langle \alpha(1) \cdots \alpha(M-1) \, | x_k^\beta | \, \alpha'(1) \cdots \alpha'(M-1) \rangle \\ &\quad = \left[\prod_{l \neq k} \delta(\alpha(l) - \alpha'(l)) \right] \delta(\alpha(k) - \beta - \alpha'(k)) \\ &\qquad \times \exp\{\pi i [B(\alpha(k) - \alpha'(k), \alpha(k+1)) - B(\alpha(k-1), \alpha(k) - \alpha'(k)) \\ &\qquad - B(\alpha(k), \alpha'(k)) + B(\alpha'(k), \alpha(k))]\} \end{aligned} \tag{4.13}$$

It can be immediately verified that the matrices x_k^α obey the following commutation relations:

$$\begin{aligned} x_k^0 &= 1 \\ x_k^\alpha x_k^\beta &= e^{\pi i A(\beta, \alpha)} x_k^{\alpha + \beta} \\ x_k^\alpha x_{k+1}^\beta &= \omega^{2\pi i B(\alpha, \beta)} x_{k+1}^\beta x_k^\alpha \\ x_k^\alpha x_{k'}^\beta &= x_{k'}^\beta x_k^\alpha \qquad \text{if} \quad |k - k'| \geqslant 2 \end{aligned} \tag{4.14}$$

These are generalizations of the cyclotomic algebra which appears in the case N finite.[8–11,7] The trace of the operators x_k^α can be defined by the following equation:

$$\mathrm{Tr}(x_k^\alpha) = \delta(\alpha) \tag{4.15}$$

By using the commutation relations (4.14), Eq. (4.5), the definitions (4.11) and (4.15) for the trace, and a suitable Seifert form (the same used in ref. 8), we find that Eq. (4.9) is immediate. Further it can be observed that the matrix Q defined in (4.10) has a precise topological meaning, because it is a presentation matrix for the module $H_1(\Sigma_n, \mathbf{Z})$ if it is interpreted a matrix with integer entries and Σ_n is the nth cyclic covering space of S^3 branched along the knot. It can be noticed that we obtain

$$\mathrm{Tr}(1) \to \infty \tag{4.16}$$

but this is related to the fact that the dimension of the space is infinite. The trace of an element of the braid group t which can be closed to give a knot K and hence the expression (4.9) is well defined whenever the matrix Q is not singular and hence the Betty number of $H_1(\Sigma_n, \mathbf{Z})$ is 0. This could be expected, because in the case N finite the absolute value of the invariant is given by N raised to the Betty number, and this diverges in the large-N limit if the Betty number is not 0.

By applying the formula to calculate a Gaussian integral, expression (4.9) becomes

$$\mathrm{Tr}(t) = \frac{1}{[i^{v(n-1)}]^{1/2}} \det(B)^{-1/2} \det(S \otimes 1 + S^T \otimes B^T B^{-1})^{-1/2} \tag{4.17}$$

The $n-1$ eigenvalues x_i, $i = 1, \ldots, n-1$, of $-B^T B^{-1}$ are exactly the $n-1$ nth roots of unity which do not coincide with 1. This can be verified with the following procedure. B^{-1} is the $n-1 \times n-1$ matrix given by

$$(B^{-1})_{ij} = \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{4.18}$$

and therefore $B^T B^{-1}$can be written as

$$(B^T B^{-1})_{ij} = \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = i+1 \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

But the characteristic polynomial of $-B^T B^{-1}$ is

$$(-1)^{n-1} \sum_{i=0}^{n-1} t^i \tag{4.20}$$

and therefore the eigenvalues of this matrix are the $n-1$ nth roots of unity which do not coincide with 1. Further, it results that

$$\det(B) = (-1)^{n-1} \tag{4.21}$$

This means that we obtain for the partition function $\mathscr{Z}_\infty$ the expression

$$\mathscr{Z}_\infty = \mathrm{Tr}(t) = \frac{1}{[i^{(n-1)v}(-1)^{n-1}]^{1/2}} \prod_{i=1}^{n-1} \Delta_K(x_i)^{-1/2} \tag{4.22}$$

Here $\Delta_K(x)$ denotes the Alexander polynomial of the knot, and in (4.22) we have used the following expression for Δ_K:

$$\Delta_K(x) = \det(S - xS^T) \tag{4.23}$$

It can be noticed that in the case $n=2$ we have only the eigenvalue $x_1 = -1$. As a consequence the module of the invariant just gives the inverse of the square root of the invariant known as the determinant of the knot, which gives the order of the Abelian group $H_1(\Sigma_2, Z)$.

It can be seen that in the particular case when n is odd, expression (4.22) can be written in the form

$$\mathscr{Z}_\infty = \mathrm{Tr}(t) = \prod_{i=1}^{(n-1)/2} x_i^{(2v-g)/2} \Delta_K(x_i)^{-1} \tag{4.24}$$

because with the normalization of the Alexander polynomial implicit in definition (4.23) we have

$$\Delta_K(x) = x^{2v-g} \Delta_K(x^{-1}) \tag{4.25}$$

if g is the degree of the Alexander polynomial, and

$$i^{v(n-1)}(-1)^{n-1} = 1 \tag{4.26}$$

because v is even for any Seifert matrix.

This means that the expression (4.24) is constructed in such a way as to give exactly a product of Alexander–Conway polynomials. Therefore (4.24) is real, because this polynomial is symmetric.

Moreover, it can be noticed that the inverse of a product of Alexander polynomials $\Delta_K(x_i)$ can be understood as the inverse of the Alexander polynomial $\Delta_L(\mathbf{x})$ for a suitable link L with components $\{K_i\}_{i=1,\ldots,(n-1)/2}$, where n is odd and the K_i are $(n-1)/2$ copies of the knot K appearing in the previous section and such that $\mathbf{x}=\{x_i \mid x_i \in H_1(S^3 - K_i)\}$. Then it is well known that Δ_L can be related[13] to the Reidemeister–Ray–Singer torsion of the manifold obtained by closing the complementary space in S^3 of L. This is a so-called Dehn surgery topological invariant of three-manifolds.

Equations (4.22) and (4.24) are our main results.

Let us conclude this section with some remarks. First, we notice that the invariants are well defined whenever the Alexander invariant of the link is not 0 for some nth root of unity.

Moreover, the same invariants can be obtained by several other spectral limits which differ from (4.6), for instance, the limits given by

$$\begin{array}{ll} \text{(Ib)} & q \ll p \ll p' = q' \\ \text{(IIa)} & p' \ll q' \ll p = q \\ \text{(IIb)} & q' \ll p' \ll p = q \end{array} \tag{4.27}$$

In the case (Ib) we recover the inverse of the operators Y_k, while in the other two cases we obtain operators which are the transposed matrices of the preceding ones.

Probably the same invariants may not be obtained by applying a different procedure to calculate the large-N limit of the model.

5. CONCLUSIONS AND GENERALIZATIONS

In this paper we have studied the free Gaussian boson model obtained as a particular large-N limit of the Baxter–Bazhanov model. We have proved that the knot invariants associated with the (infinite-dimensional) braid group representation arising therefrom can be expressed as products of Alexander invariants (4.22), (4.24).

As the cyclotomic invariants which are obtained in the case N finite are related to the module $H_1(\Sigma_n, \mathbf{Z}_N)$, where Σ_n is the nth cyclic covering of the knot with which the invariant is associated, it would also be interesting to investigate the limit $n \to \infty$ of the model, in which we should obtain something related to the infinite cylic covering of the knot. But the large-n limit is interesting for other reasons, too. In the first place, it corresponds to the thermodynamic limit and there are hints that the Baxter–Bazhanov model in that limit is relted to a parafermionic model.[21] In the second place, we would obtain informations about $U_q(sl(n))$ when $n \to \infty$.

Another observation is that in the so-called large-coupling-constant limit, the dominant part of the partition function of a three-dimensional continuum non-Abelian $SU(2)$ Chern–Simons theory over a three-manifold M^3 obtained by Dehn surgery along a link L in a homological sphere is proportional to the inverse of the Alexander polynomial of L.[22] It would be interesting to determine if there is some deeper reason these partition functions coincide.

We hope to return to these themes in a forthcoming work.

REFERENCES

1. V. V. Bazhanov and R. J. Baxter, New solvable lattice models in three dimensions, *J. Stat. Phys.* **69**:453 (1992).
2. A. B. Zamolodchikov, Tetrahedra equations and integrable systems in three-dimensional space, *Zh. Eksp. Teor. Fiz.* **79**:641 (1980) [*Sov. Phys. JETP* **52**:325 (1980)].
3. A. B. Zamolodchikov, Tetrahedron equations and the relativistic S-matrix of straight-strings in 2 + 1-dimensions, *Commun. Math. Phys.* **79**:489 (1981).
4. V. V. Bazhanov, R. M. Kashaev, V. V. Mangazeev, and Yu. G. Stroganov, Z_N^{n-1} generalization of the chiral Potts model, *Commun. Math. Phys.* **138**:393 (1991).
5. E. Date, M. Jimbo, K. Miki, and T. Miwa, Generalized chiral Potts models and minimal cyclic representations of $U_q(\hat{gl}(n, C))$, *Commun. Math. Phys.* **137**:133 (1991).
6. R. M. Kashaev, V. V. Mangazeev, and T. Nakanishi, Yang–Baxter equation for the $sl(n)$ chiral Potts model, *Nucl. Phys. B* **362**:563 (1991).
7. B. L. Cerchiai, M. Martellini, and F. Valz-Gris, Three-dimensional integrable models and associated tangle invariants, *J. Stat. Phys.* **78**:1083 (1994).
8. E. Date, M. Jimbo, K. Miki, and T. Miwa, Braid group representations arising from the generalized chiral Potts models, *Pac. J. Math.* **154**:37 (1992).
9. T. Kobaiashi, H. Murakami, and J. Murakami, Cyclotomic invariants for links, *Proc. Jpn. Acad.* **64A**:235 (1988).
10. V. R. Jones, Baxterization, *Int. J. Mod. Phys. A* **6**:2035 (1991).
11. D. Goldschmidt and V. R. Jones, Metaplectic link invariants, *Geom. Dedicata* **31**:165 (1989).
12. V. V. Bazhnov and R. J. Baxter, Star–triangle relation for a three dimensional model, *J. Stat. Phys.* **71**:839 (1993).
13. V. Turaev, *Russ. Math. Surv.* **41**:199 (1986).
14. R. J. Baxter, On Zamolodchikov's solution of the tetrahedron equations, *Commun. Math. Phys.* **88**:185 (1983).
15. R. M. Kashaev, V. V. Mangazeev, and Yu. G. Stroganov, Spatial symmetry, local integrability and tetrahedron equations in the Baxter–Bazhanov model, *Int. J. Mod. Phys. A* **8**:587 (1993).
16. R. M. Kashaev, V. V. Mangazeev, and Yu. G. Stroganov, Star–square and tetrahedron equations in the Baxter–Bazhanov model, *Int. J. Mod. Phys. A* **8**:1399 (1993).
17. T. Deguchi, M. Wadati, and Y. Akutsu, Knot theory based on solvable models at criticality, *Adv. Stud. Pure Math.* **19**:193 (1989).
18. M. Wadati, T. Deguchi, and Y. Akutsu, Exactly solvable models and knot theory, *Phys. Rep.* **180**:247 (1989).

19. V. A. Fateev and A. B. Zamolodchikov, Self-dual solutions of the star–triangle relations in $\mathbf{Z}_N$-models, *Phys. Lett.* **92A**:37 (1982).
20. V. A. Fateev and A. B. Zamolodchikov, The exactly solved case of a 2D lattice of plane rotators, *Phys. Lett.* **92A**:35 (1982).
21. D. Gepner, *Nucl. Phys. B* **290**[FS20]:10 (1987).
22. L. Rozansky, A contribution of the trivial connection to the Jones polynomial and Witten's invariant of 3d manifolds, II, preprint UMTG-176-94, hepth/9403021 (1994).